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ON THE INSTABILITY OF A PARTICULAR THIRD-ORDER LINEAR SYSTEM

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We consider a differential equation of the form

$$x^{\cdot\cdot\cdot} + p(t)\ x = 0 \tag{1}$$

where function $p(t)$ is continuous and ω-periodic. Equation (1) is called stable or unstable depending on whether the related system is stable or unstable.

Sufficient conditions of instability of Eq. (1) are derived with the use of Liapunov's construction which is utilized for proving the stability of equation $x^{\cdot\cdot} + p(t)\ x = 0$ (see [1]).

We denote by $z_k(t)$ $(k = 1,2,3)$ the linearly independent solutions of Eq. (1) which satisfy the initial conditions

$$z_k(0) = \delta_{k1}, \quad z_k^{\cdot}(0) = \delta_{k2}, \quad z_k^{\cdot\cdot}(0) = \delta_{k3}, \quad k = 1,2,3$$

where $\delta_{k,l}$ is the Kronecker delta, and consider its monodromy matrix [1]. For the determination of multipliers we then obtain the equation

$$\rho^3 - a\rho^2 + b\rho - 1 = 0 \tag{2}$$

$$a = z_1(\omega) + z_2^{\cdot}(\omega) + z_3^{\cdot\cdot}(\omega), \quad b = [z_1(\omega) z_3^{\cdot\cdot}(\omega) - z_3(\omega)\ z_1^{\cdot\cdot}(\omega)] +$$
$$[z_1(\omega)\ z_2^{\cdot}(\omega) - z_2(\omega)\ z_1^{\cdot}(\omega)] + [z_2^{\cdot}(\omega)\ z_3^{\cdot\cdot}(\omega) - z_3^{\cdot}(\omega)\ z_2^{\cdot\cdot}(\omega)]$$

As in [1], we have

$$z_k(t) = \left(1 - \frac{1}{2}\delta_{k3}\right)\left\{t^{k-1} - \frac{1}{2}\int_0^t p(t_1)(t-t_1)^2 t_1^{k-1} dt_1 + \right. \tag{3}$$
$$\left. \frac{1}{4}\int_0^t p(t_1)\left[p(t_2)(t_1 - t_2)^2 t_2^{k-1} dt_2\right]\ (t-t_1)^2\ dt_1 - \ldots\right\}$$

Introducing into the analysis the numerical series

$$\sum_{k=1}^{\infty}(q/3!)^k \quad (q = M\omega^3, |\,p(t)\,| \leqslant M)$$

which is convergent for $q < 6$ and whose sum S is equal $q/(6-q))$, after some simple transformations, we obtain the estimates

$$z_1^{\cdot}(\omega) \leqslant M\ \omega^2\ (S+1), \qquad z_1^{\cdot\cdot}(\omega) \leqslant M\omega\ (S+1) \tag{4}$$
$$z_2(\omega) \leqslant \omega\ (S+1), \qquad z_2^{\cdot\cdot}(\omega) \leqslant M\ \omega^2\ (S+1)$$
$$z_3(\omega) \leqslant {}^1/_2\ \omega^2 + \omega^2 S, \qquad z_3^{\cdot}(\omega) \leqslant \omega\ (S+1)$$

L e m m a. Let $p(t) \not\equiv 0$ and let the inequalities

$$p(t) \leqslant 0, \quad q = M\omega^3 < {}^1/_2\,(75 - 9\sqrt{65}) = d \quad (1 < d < 2) \tag{5}$$

be satisfied. Then

$$a + b + 2 > 0 \tag{6}$$

Proof. Formulas (3) and the definition of the number a imply that $a > 3$. From the inequality (4) we obtain

$$z_3^{\cdot}(\omega)\, z_2^{\cdot\cdot}(\omega) \leqslant q\,(S+1)^2, \qquad z_2(\omega)\, z_1^{\cdot}(\omega) \leqslant q\,(S+1)^2$$
$$z_3^{\cdot}(\omega)\, z_1^{\cdot\cdot}(\omega) \leqslant q\,(S+1)\,(S+{}^1/_2)$$

From this and the positiveness of solutions (3) and of their first and second derivatives we find that for $t = \omega$

$$b \geqslant -q^*, \qquad q^* = q\,(S+1)\,(3S+{}^5/_2)$$

To complete the proof of the lemma it remains to show that $q^* < 5$ or

$$q^2 - 75q + 90 > 0 \tag{7}$$

Inequality (7) follows from condition (5) of the lemma, since the number d is the smaller root of the trinomial in the left-hand part of (7). The lemma is proved.

If $a \neq b$, the number $\rho = 1$ is not a root of Eq. (2). Substituting into (2) $\rho = (\lambda + 1) / (\lambda - 1)$, we obtain the equation

$$(b-a)\,\lambda^3 + (6-a-b)\,\lambda^2 + (a-b)\,\lambda + (a+b+2) = 0 \tag{8}$$

for which $\lambda = 1$ is not a root.

Theorem. If the continuous and ω-periodic function $p(t)$ is nonpositive and $p(t) \not\equiv 0$ and inequality (5) is satisfied, then Eq. (1) is unstable.

Proof. Let initially $a \neq b$. Then by virtue of condition (6) the polynomial in the left-hand part of Eq. (8) is a standard polynomial [1]. It follows from Hurwitz conditions that Eq. (8) has at least one root in the right-hand half-plane. Reverting to the variable ρ, we conclude that Eq. (1) has at least one multiplier ρ for which $|\rho| > 1$. This proves the theorem for $a \neq b$. When $a = b$ Eq. (2) has the root

$$\rho = {}^1/_2\,(a - 1 + \sqrt{a^2 + 2a - 3}) > 1 \qquad (a > 3)$$

Hence Eq. (1) is unstable even for $a = b$.

The theorem is proved.

REFERENCE

1. Demidovich, B. P., Course of Mathematical Theory of Stability. "Nauka", Moscow, 1967.

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